Variable Eddington Factor Acceleration of Thermal Radiative Transfer on Curved Meshes

HEDP End of Summer Presentation

Samuel S. Olivier[1]

Mentors: Peter Maginot[2], Terry Haut[2]

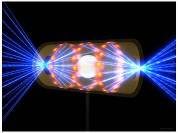
August 15, 2018

[1]University of California, Berkeley

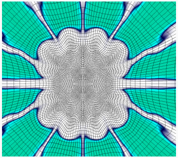
[2]Lawrence Livermore National Laboratory

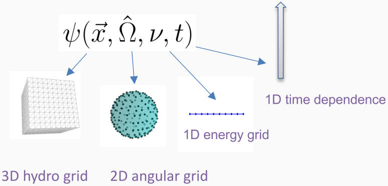
LLNL-PRES-757684

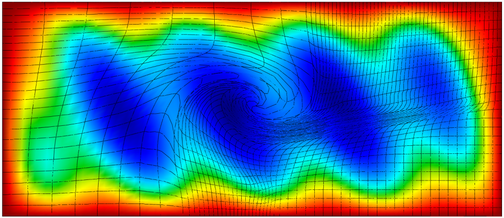
Motivation



- Thermal Radiative Transfer: describes conservation and transfer of energy between photons and matter
- 6+1 dimensional phase space $\Rightarrow$ dominates memory and runtime
- LDRD investigating high order FEM TRT on curved meshes for coupling to BLAST hydrodynamics code
- Goal: develop an acceleration scheme to improve iterative convergence







Linear Transport Equation

- Steady state, one-group, Linear Boltzmann Equation with isotropic scattering and source:

$$\hat{\Omega} \cdot \nabla \psi + \sigma_t \psi = \frac{\sigma_s}{4\pi} \int \psi \, \mathrm{d}\Omega + \frac{Q}{4\pi}$$

Streaming Collision Scattering Source

- Discrete Ordinates ($\mathrm{S_N}$) angular discretization

$$\hat{\Omega} \cdot \nabla \psi_d + \sigma_t \psi_d = \frac{\sigma_s}{4\pi} \sum w_{d'} \psi_{d'} + \frac{Q}{4\pi} \,, \quad d = 1, 2, \ldots, N_{\text{angles}}$$

where $\psi_d(x) = \psi(x, \hat{\Omega}_d)$

- N_{angles} coupled equations $\Rightarrow$ prohibitively expensive to solve

Source Iteration decouples in angle

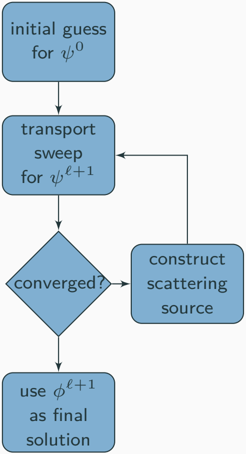
- Decouple by lagging the scattering term

$$\hat{\Omega} \cdot \nabla \psi_d^{\ell+1} + \sigma_t \psi_d^{\ell+1} = \frac{\sigma_s}{4\pi} \sum w_{d'} \psi_{d'}^{\ell} + \frac{Q}{4\pi}$$

Known from previous iteration

$\rightarrow N_{\text{angles}}$ independent equations but need to solve iteratively

- Slow convergence in highly scattering systems



Need a preconditioner/accelerator for practical applications

Variable Eddington Factor Equations

- Take first two angular moments of transport equation:

$$\nabla \cdot \vec{J} + \sigma_a \phi = Q$$

$$\nabla \cdot (\boldsymbol{E}\phi) + \sigma_t \vec{J} = 0$$

where

$$\phi = \int \psi \, \mathrm{d}\Omega \,, \quad \vec{J} = \int \hat{\Omega} \, \psi \, \mathrm{d}\Omega$$

and

$$\boldsymbol{E} = \frac{\int \hat{\Omega} \otimes \hat{\Omega} \, \psi \, \mathrm{d}\Omega}{\int \psi \, \mathrm{d}\Omega}$$

- In 3D: 4 equations for 13 unknowns
- More angular moments $\rightarrow$ more unknowns
- Need ψ (the solution) to have closure
- Historically: invent a closure similar to flux limited diffusion

Close with transport information from previous iteration

- Solve

$$\hat{\Omega} \cdot \nabla \psi^{\ell+1/2} + \sigma_t \psi^{\ell+1/2} = \frac{\sigma_s}{4\pi} \phi^\ell + \frac{Q}{4\pi}$$

  for $\psi^{\ell+1/2}$

- Compute Eddington tensor:

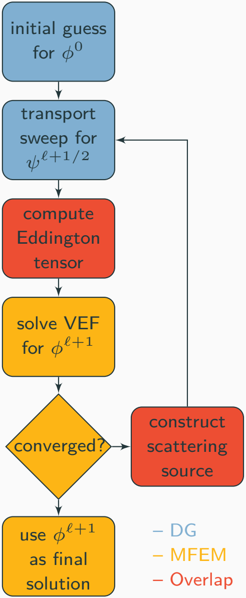
$$\boldsymbol{E}_{ij}^{\ell+1/2} = \frac{\sum w_d \hat{\Omega}_i^{(d)} \hat{\Omega}_j^{(d)} \psi_d^{\ell+1/2}}{\sum w_d \psi_d^{\ell+1/2}}$$

- Solve VEF equations for updated scalar flux $\phi^{\ell+1}$

$$\nabla \cdot \vec{J}^{\ell+1} + \sigma_a \phi^{\ell+1} = Q \,,$$

$$\nabla \cdot \left(\boldsymbol{E}^{\ell+1/2} \phi^{\ell+1} \right) + \sigma_t \vec{J}^{\ell+1} = 0 \,.$$

- Update scattering term with VEF solution
- Stop if $\|\phi^{\ell+1} - \phi^\ell\| < tol$



VEF Accelerates Source Iteration

- Eddington tensor converges faster than the scalar flux
  - Angular flux weighted average of $\hat{\Omega} \otimes \hat{\Omega} \Rightarrow$ depends on angular shape not magnitude
  - ψ converges quickly in angular shape
- VEF compensates lagging of scattering term in Source Iteration

Mixed Finite Element Discretization

$$\nabla \cdot \vec{J} + \sigma_a \phi = Q$$
$$\nabla \cdot (\boldsymbol{E}\phi) + \sigma_t \vec{J} = 0$$

VEF discretization does not need to match transport!

Mixed Finite Element:

- $\vec{J}$ with $H^{1,d}$ finite elements (vector lagrange)
- ϕ with L^2 finite elements (discontinuous)

Multiply zeroth moment by ϕ basis function, u, and integrate:

$$\int u \nabla \cdot \vec{J}_h \, \mathrm{d}V + \int \sigma_a u \phi_h \, \mathrm{d}V = \int u Q \, \mathrm{d}V$$

Multiply first moment by $\vec{J}$ basis function, $\vec{v}$, and integrate tensor term by parts:

$$\int \phi_h \boldsymbol{E} : \nabla \vec{v} \, \mathrm{d}V - \int \sigma_t \vec{v} \cdot \vec{J}_h \, \mathrm{d}V = \int_{\partial V} \boldsymbol{E}\hat{n} \cdot \vec{v} \phi_h^{\mathsf{BC}} \, \mathrm{d}S$$

MFEM leads to non-symmetric Saddle Point problem

Matrix form:

$$\begin{bmatrix} \mathbf{M}_a & \mathbf{G} \\ \mathbf{H} & -\mathbf{M}_t \end{bmatrix} \begin{bmatrix} \underline{\phi} \\ \underline{J} \end{bmatrix} = \begin{bmatrix} \underline{Q} \\ \underline{B} \end{bmatrix}$$

where

$$\begin{aligned} \mathbf{M}_{a,ij} &= \int \sigma_a u_i u_j \,\mathrm{d}V \,, & \mathbf{G}_{ij} &= \int u_i \nabla \cdot \vec{v}_j \,\mathrm{d}V \,, \\ \mathbf{H}_{ij} &= \int u_j \boldsymbol{E} : \nabla \vec{v}_i \,\mathrm{d}V \,, & \mathbf{M}_{t,ij} &= \int \sigma_t \vec{v}_i \cdot \vec{v}_j \,\mathrm{d}V \\ \underline{Q}_i &= \int u_i Q \,\mathrm{d}V \,, & \underline{B}_i &= \int_{\partial V} \boldsymbol{E}\hat{n} \cdot \vec{v}_i \phi_h^{\mathsf{BC}} \,\mathrm{d}S \end{aligned}$$

$\mathbf{H} \neq \mathbf{G}^T$ due to presence of Eddington Tensor $\Rightarrow$ non-symmetric Saddle Point Problem

Solve with Schur Complement

$$\begin{bmatrix} \mathbf{M}_a & \mathbf{G} \\ \mathbf{H} & -\mathbf{M}_t \end{bmatrix} \begin{bmatrix} \underline{\phi} \\ \underline{J} \end{bmatrix} = \begin{bmatrix} \underline{Q} \\ \underline{B} \end{bmatrix}$$

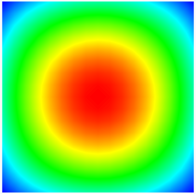
- $\mathbf{M}_a$ is block diagonal $\Rightarrow$ easily inverted!

$$\mathbf{M}_a \underline{\phi} + \mathbf{G}\underline{J} = \underline{Q}$$
$$\Rightarrow \underline{\phi} = \mathbf{M}_a^{-1} \left[\underline{Q} - \mathbf{G}\underline{J} \right]$$

$$\mathbf{H}\underline{\phi} - \mathbf{M}_t \underline{J} = \underline{B}$$
$$\Rightarrow -\underbrace{\left[\mathbf{H}\mathbf{M}_a^{-1}\mathbf{G} + \mathbf{M}_t \right]}_{\text{Schur Complement}=\mathbf{S}} \underline{J} = \underline{B} - \mathbf{H}\mathbf{M}_a^{-1}\underline{Q}$$

- Assemble and solve at every iteration $\Rightarrow$ want iterative solver
- Schur Complement is non-symmetric and has been difficult to solve iteratively (need to find a preconditioner)

Method of Manufactured Solutions Test Problem



Geometry: 2D box $[0,1] \times [0,1]$

$\sigma_t = 5\,\mathrm{cm}^{-1} \Rightarrow$ 5 mfp thick

Set the fixed source to force the solution to "chopped sines"

$$\phi = \sin\left(\pi \frac{x+\alpha}{1+2\alpha}\right) \sin\left(\pi \frac{y+\alpha}{1+2\alpha}\right)$$

$\alpha \neq 0$ allows testing inflow boundary conditions

Provides known solution to compare numerical solution to

S_4 angular quadrature

VEF Accelerates Source Iteration

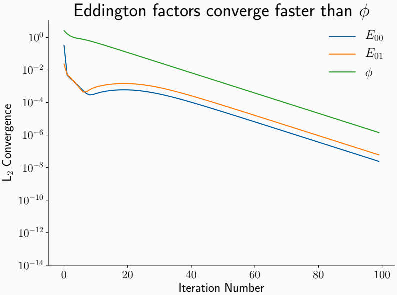


Fig. 1: Convergence for unaccelerated Source Iteration.

VEF Accelerates Source Iteration

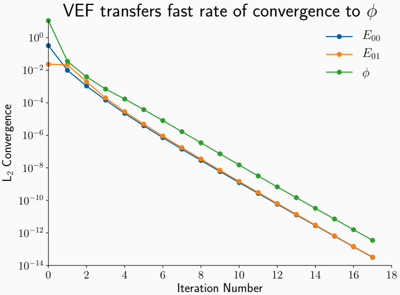


Fig. 2: Convergence for VEF acceleration.

VEF Accelerates Source Iteration

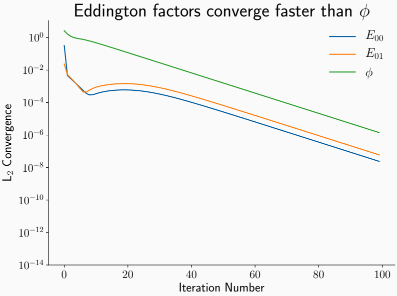


Fig. 1: Convergence for unaccelerated Source Iteration.

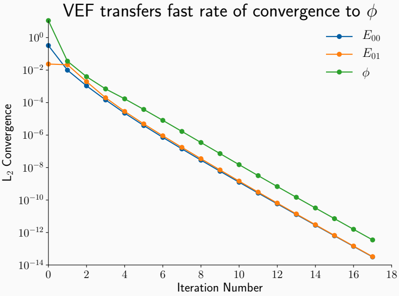
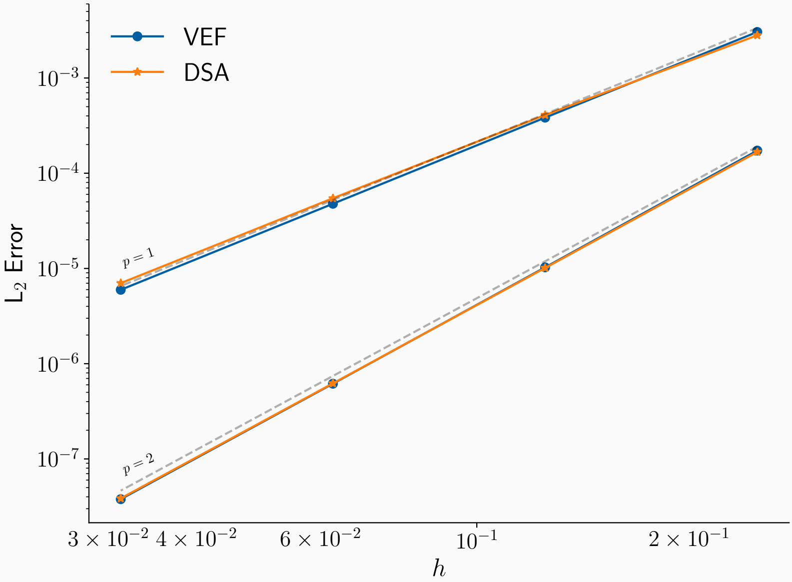
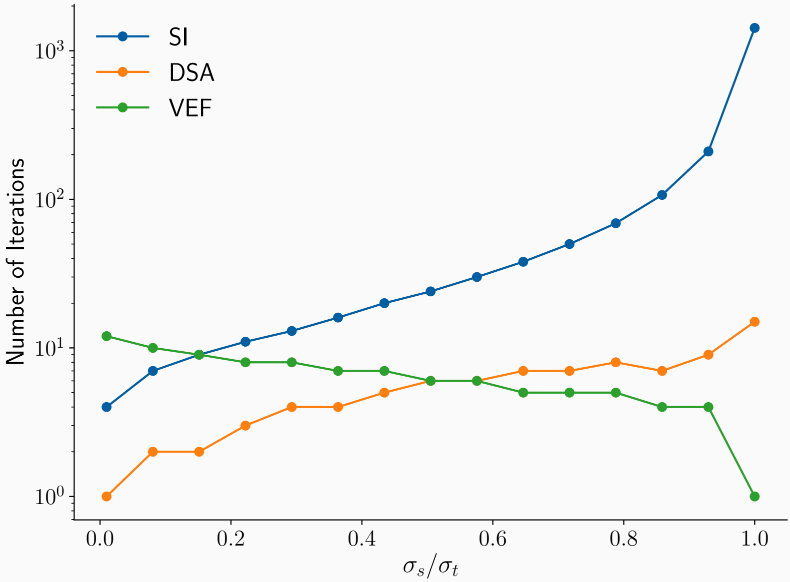


Fig. 2: Convergence for VEF acceleration.

VEF algorithm maintains order of accuracy



VEF accelerates as well as traditional methods



Conclusions

- Developed a mixed finite element discretization for Eddington equations + a robust Saddle Point solver
- Implemented VEF acceleration in LDRD code
- Verified order of accuracy and acceleration properties

References

[1] *MFEM: Modular finite element methods library*. `mfem.org`.

[2] V. Y. Gol'din, *A quasi-diffusion method of solving the kinetic equation*, USSR Comp. Math. and Math. Physics, 4 (1964), pp. 136–149.

[3] P. G. Maginot and T. A. Brunner, *Lumping techniques for mixed finite element diffusion discretizations*, Journal of Computational and Theoretical Transport, 0 (2018), pp. 1–25.

[4] M. Miften and E. Larsen, *The quasi-diffusion method for solving transport problems in planar and spherical geometries*, Journal of Transport Theory and Statistical Physics, 22(2-3) (1992), pp. 165–186.

[5] S. S. Olivier and J. E. Morel, *Variable eddington factor method for the sn equations with lumped discontinuous galerkin spatial discretization coupled to a drift-diffusion acceleration equation with mixed finite-element discretization*, Journal of Computational and Theoretical Transport, 46 (2017), pp. 480–496.

Thanks to my mentors Peter Maginot and Terry Haut.

Lawrence Livermore National Laboratory