Variable Eddington Factor Acceleration of Thermal Radiative Transfer on Curved Meshes

HEDP End of Summer Presentation

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Motivation

- Thermal Radiative Transfer: describes conservation and transfer of energy between photons and matter
- 6+1 dimensional phase space ⇒ dominates memory and runtime
- LDRD investigating high order FEM TRT on curved meshes for coupling to BLAST hydrodynamics code
- Goal: develop an acceleration scheme to improve iterative convergence

\[ \psi(\vec{x}, \hat{\Omega}, \nu, t) \]

3D hydro grid 2D angular grid

1D time dependence 1D energy grid
Linear Transport Equation

- Steady state, one-group, Linear Boltzmann Equation with isotropic scattering and source:

\[ \hat{\Omega} \cdot \nabla \psi + \sigma_t \psi = \frac{\sigma_s}{4\pi} \int \psi \, d\Omega + \frac{Q}{4\pi} \]

- Discrete Ordinates ($S_N$) angular discretization

\[ \hat{\Omega} \cdot \nabla \psi_d + \sigma_t \psi_d = \frac{\sigma_s}{4\pi} \sum w_{d'} \psi_{d'} + \frac{Q}{4\pi}, \quad d = 1, 2, \ldots, N_{\text{angles}} \]

where \( \psi_d(x) = \psi(x, \hat{\Omega}_d) \)

- \( N_{\text{angles}} \) coupled equations \( \Rightarrow \) prohibitively expensive to solve
Source Iteration decouples in angle

- Decouple by lagging the scattering term

\[ \hat{\Omega} \cdot \nabla \psi_{d}^{\ell+1} + \sigma_{t} \psi_{d}^{\ell+1} = \frac{\sigma_{s}}{4\pi} \sum w_{d'} \psi_{d'}^{\ell} + \frac{Q}{4\pi} \]

Known from previous iteration

\[ \rightarrow N_{\text{angles}} \] independent equations but need to solve iteratively

- Slow convergence in highly scattering systems

Need a preconditioner/accelerator for practical applications
Variable Eddington Factor Equations

- Take first two angular moments of transport equation:

\[ \nabla \cdot \vec{J} + \sigma_a \phi = Q \]

\[ \nabla \cdot (E \phi) + \sigma_t \vec{J} = 0 \]

where

\[ \phi = \int \psi \, d\Omega, \quad \vec{J} = \int \hat{\Omega} \, \psi \, d\Omega \]

and

\[ E = \frac{\int \hat{\Omega} \otimes \hat{\Omega} \, \psi \, d\Omega}{\int \psi \, d\Omega} \]

- In 3D: 4 equations for 13 unknowns
- More angular moments \(\rightarrow\) more unknowns
- Need \(\psi\) (the solution) to have closure
- Historically: invent a closure similar to flux limited diffusion
Close with transport information from previous iteration

- Solve
  \[ \hat{\Omega} \cdot \nabla \psi_{\ell+1/2} + \sigma_t \psi_{\ell+1/2} = \frac{\sigma_s}{4\pi} \phi^\ell + \frac{Q}{4\pi} \]
  for \( \psi_{\ell+1/2} \)

- Compute Eddington tensor:
  \[ E_{ij}^{\ell+1/2} = \frac{\sum w_d \hat{\Omega}_i^{(d)} \hat{\Omega}_j^{(d)} \psi_d^{\ell+1/2}}{\sum w_d \psi_d^{\ell+1/2}} \]

- Solve VEF equations for updated scalar flux \( \phi^{\ell+1} \)
  \[ \nabla \cdot \vec{J}^{\ell+1} + \sigma_a \phi^{\ell+1} = Q, \]
  \[ \nabla \cdot \left( E^{\ell+1/2} \phi^{\ell+1} \right) + \sigma_t \vec{J}^{\ell+1} = 0. \]

- Update scattering term with VEF solution

- Stop if \( \| \phi^{\ell+1} - \phi^\ell \| < tol \)
VEF Accelerates Source Iteration

- Eddington tensor converges faster than the scalar flux
  - Angular flux weighted average of $\hat{\Omega} \otimes \hat{\Omega}$ depends on angular shape not magnitude
  - $\psi$ converges quickly in angular shape
- VEF compensates lagging of scattering term in Source Iteration
Mixed Finite Element Discretization

VEF discretization does not need to match transport!

Mixed Finite Element:

- \( \vec{J} \) with \( H^{1,d} \) finite elements (vector lagrange)
- \( \phi \) with \( L^2 \) finite elements (discontinuous)

Multiply zeroth moment by \( \phi \) basis function, \( u \), and integrate:

\[
\int u \nabla \cdot \vec{J}_h \, dV + \int \sigma_a u \phi_h \, dV = \int uQ \, dV
\]

Multiply first moment by \( \vec{J} \) basis function, \( \vec{v} \), and integrate tensor term by parts:

\[
\int \phi_h E : \nabla \vec{v} \, dV - \int \sigma_t \vec{v} \cdot \vec{J}_{h} \, dV = \int_{\partial\Omega} E \hat{n} \cdot \vec{v} \phi_{h}^{BC} \, dS
\]

\[\nabla \cdot \vec{J} + \sigma_a \phi = Q \]

\[\nabla \cdot (E \phi) + \sigma_t \vec{J} = 0\]
MFEM leads to non-symmetric Saddle Point problem

Matrix form:

\[
\begin{bmatrix}
M_a & G \\
H & -M_t
\end{bmatrix}
\begin{bmatrix}
\phi \\
J
\end{bmatrix}
=
\begin{bmatrix}
Q \\
B
\end{bmatrix}
\]

where

\[
M_{a,ij} = \int \sigma_a u_i u_j \, dV ,
\]
\[
G_{ij} = \int u_i \nabla \cdot \vec{v}_j \, dV ,
\]
\[
H_{ij} = \int u_j \mathbf{E} : \nabla \vec{v}_i \, dV ,
\]
\[
M_{t,ij} = \int \sigma_t \vec{v}_i \cdot \vec{v}_j \, dV
\]
\[
Q_i = \int u_i Q \, dV ,
\]
\[
B_i = \int_{\partial V} \mathbf{E} \hat{n} \cdot \vec{v}_i \phi_{h}^{BC} \, dS
\]

\( H \neq G^T \) due to presence of Eddington Tensor \( \Rightarrow \) non-symmetric Saddle Point Problem
Solve with Schur Complement

- $M_a$ is block diagonal $\Rightarrow$ easily inverted!

\[ M_a \phi + G J = Q \]

$\Rightarrow \phi = M_a^{-1} [Q - G J]$ 

\[ H \phi - M_t J = B \]

$\Rightarrow - \left[ H M_a^{-1} G + M_t \right] J = B - H M_a^{-1} Q$

- Assemble and solve at every iteration $\Rightarrow$ want iterative solver

- Schur Complement is non-symmetric and has been difficult to solve iteratively (need to find a preconditioner)
Method of Manufactured Solutions Test Problem

Geometry: 2D box $[0, 1] \times [0, 1]$

$s_\tau = 5 \text{ cm}^{-1} \Rightarrow 5 \text{ mfp thick}$

Set the fixed source to force the solution to “chopped sines”

$$\phi = \sin \left( \pi \frac{x + \alpha}{1 + 2\alpha} \right) \sin \left( \pi \frac{y + \alpha}{1 + 2\alpha} \right)$$

$\alpha \neq 0$ allows testing inflow boundary conditions

Provides known solution to compare numerical solution to $S_4$ angular quadrature
VEF Accelerates Source Iteration

**Eddington factors converge faster than $\phi$**

![Graph showing convergence](image)

Fig. 1: Convergence for unaccelerated Source Iteration.
VEF Accelerates Source Iteration

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VEF transfers fast rate of convergence to $\phi$

Fig. 2: Convergence for VEF acceleration.
VEF Accelerates Source Iteration

Eddington factors converge faster than $\phi$

VEF transfers fast rate of convergence to $\phi$

Fig. 1: Convergence for unaccelerated Source Iteration.

Fig. 2: Convergence for VEF acceleration.
VEF algorithm maintains order of accuracy.

\[
\begin{align*}
10^{-1} &\times 10^{-2} & 4 \times 10^{-2} & 6 \times 10^{-2} & 2 \times 10^{-1} \\
\hbar & 10^{-7} & 10^{-6} & 10^{-5} & 10^{-4} \\
L_2 \text{ Error} \quad p = 1 & & & & \ \\
VEF & & & & \ \\
DSA & & & & \\
\end{align*}
\]
VEF accelerates as well as traditional methods.
Conclusions

- Developed a mixed finite element discretization for Eddington equations + a robust Saddle Point solver
- Implemented VEF acceleration in LDRD code
- Verified order of accuracy and acceleration properties


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