High-Order Variable Eddington Factor Methods for Thermal Radiative Transfer

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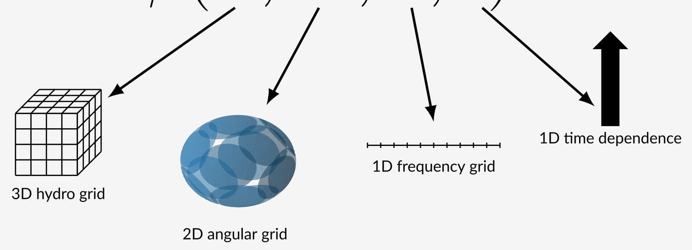
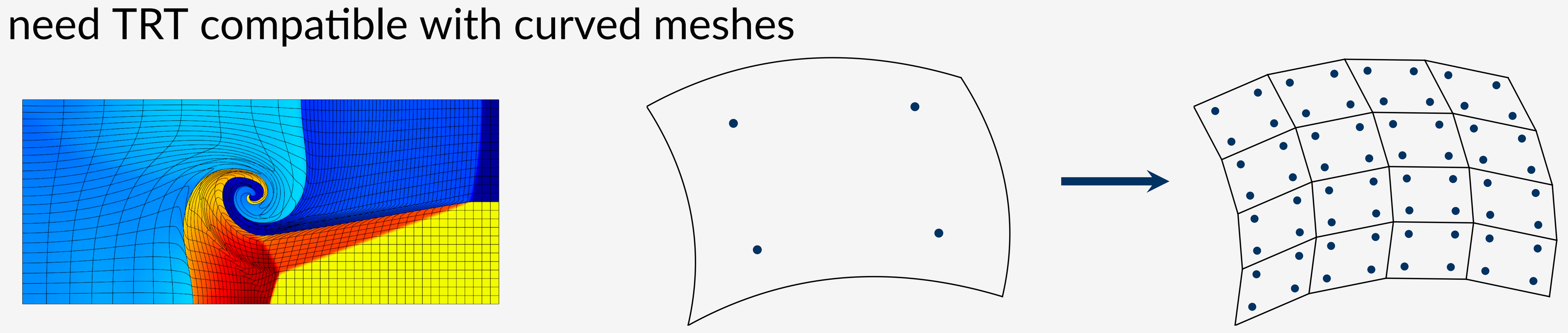
Motivation

- Model conservation of energy between photons and matter
- High energy density physics simulations requires tightly coupled modeling of hydrodynamics and radiative transfer
- TRT typically bottleneck, can be 90% of runtime and memory

$$\psi(\mathbf{x}, \mathbf{\Omega}, \nu, t)$$



- High-order hydrodynamics on curved meshes, low-order refined too expensive $\Rightarrow$ need TRT compatible with curved meshes



- Variable Eddington Factor (VEF) method: efficient transport scheme
  - Enables significant *algorithmic flexibility* not possible with traditional transport methods
  - Development of discretizations with corresponding scalable linear solvers difficult
- Goal: develop an efficient high-order Variable Eddington Factor method compatible with curved meshes

Transport Background

- Steady-state, mono-energetic, linear transport with isotropic scattering and source

$$\mathbf{\Omega} \cdot \nabla \psi + \sigma_t \psi = \frac{\sigma_s}{4\pi} \int \psi \, \mathrm{d}\Omega' + q$$

- S_N angular model: collocate at *discrete angles* chosen from a quadrature rule on the unit sphere

$$\psi_d(\mathbf{x}) = \psi(\mathbf{x}, \mathbf{\Omega}_d) \, , \quad \int \psi \, \mathrm{d}\Omega \to \sum_d w_d \psi_d$$

- Decouple in angle by lagging scattering
  - Required for problem to be computationally tractable
  - Can converge arbitrarily slowly in problems with optically thick materials

The Variable Eddington Factor Method

- Solve the coupled transport-VEF system

$$\mathbf{\Omega} \cdot \nabla \psi + \sigma_t \psi = \frac{\sigma_s}{4\pi} \varphi + q$$
$$\nabla \cdot \mathbf{J} + \sigma_a \varphi = Q_0$$
$$\nabla \cdot (\mathbf{E}\varphi) + \sigma_t \mathbf{J} = 0$$

where φ and $\mathbf{J}$ are the zeroth and first moments of ψ, $Q_0 = \int q \, \mathrm{d}\Omega$, and

$$\mathbf{E}(\mathbf{x}) = \frac{\int \mathbf{\Omega} \otimes \mathbf{\Omega} \, \psi \, \mathrm{d}\Omega}{\int \psi \, \mathrm{d}\Omega}$$

is the *Eddington tensor*

- Linear into nonlinear, added unknowns. $\mathbf{E}$ weak function of $\psi \Rightarrow$ converges rapidly

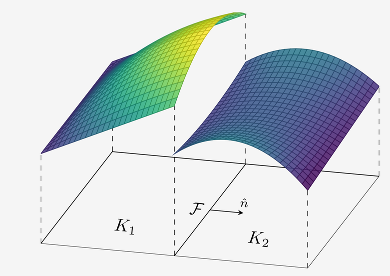
$$\varphi$$

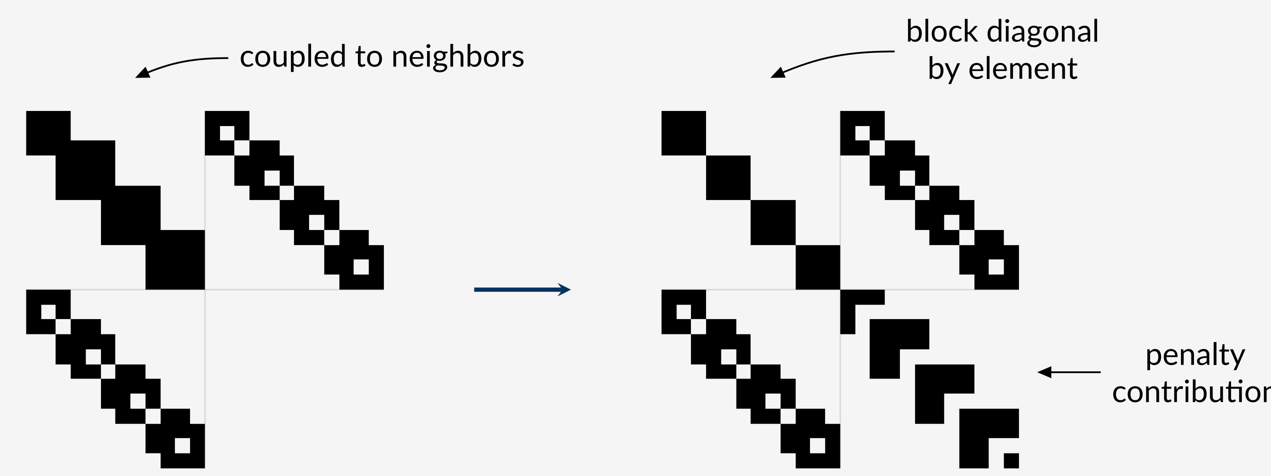
$$\mathbf{\Omega} \cdot \nabla \psi + \sigma_t \psi = \frac{\sigma_s}{4\pi} \varphi + q \qquad \begin{aligned} \nabla \cdot \mathbf{J} + \sigma_a \varphi &= Q_0 \\ \nabla \cdot (\mathbf{E}\varphi) + \sigma_t \mathbf{J} &= 0 \end{aligned}$$

$$\mathbf{E}(\cdot) = \frac{\int \mathbf{\Omega} \otimes \mathbf{\Omega} \, (\cdot) \, \mathrm{d}\Omega}{\int (\cdot) \, \mathrm{d}\Omega}$$

Discontinuous Galerkin VEF Discretization

- Represent solution and test space with piecewise *discontinuous* polynomials
- Direct discretization yields coupled 2×2 block system
- Can choose "numerical fluxes" to allow *local* elimination of the vector variable, stabilize this approximation with addition of penalty term





- Careful modifications of "Unified Analysis of DG Methods" leads to

$$\int_\Gamma \kappa \, [\![u]\!] \, [\![\varphi]\!] \, \mathrm{d}s - \int_\Gamma [\![u]\!] \left\{\!\!\left\{ \frac{1}{\sigma_t} \nabla_h \cdot (\mathbf{E}\varphi) \cdot \hat{n} \right\}\!\!\right\} \mathrm{d}s - \int_\Gamma \left\{\!\!\left\{ \frac{\nabla_h u}{\sigma_t} \right\}\!\!\right\} \cdot [\![\mathbf{E}\varphi \hat{n}]\!] \, \mathrm{d}s$$
$$+ \int \nabla_h u \cdot \frac{1}{\sigma_t} \nabla_h \cdot (\mathbf{E}\varphi) \, \mathrm{d}\mathbf{x} + \int \sigma_a \, u \varphi \, \mathrm{d}\mathbf{x} = \int u \, Q_0 \, \mathrm{d}\mathbf{x}$$

where

$$\nabla_h \cdot (\mathbf{E}\varphi) = \mathbf{E} \nabla_h \varphi + (\nabla_h \cdot \mathbf{E}) \varphi \, , \quad \nabla_h \cdot \mathbf{E} = \frac{\nabla_h \cdot \mathbf{P}\phi - \mathbf{P} \nabla_h \phi}{\phi^2}$$

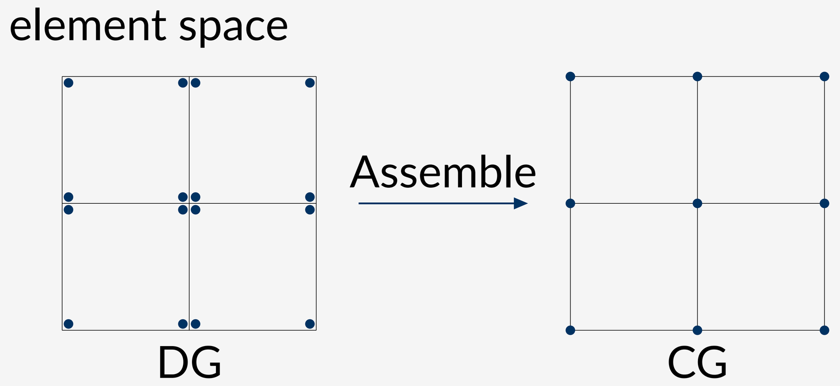
Subspace Correction Preconditioner

- Can decompose DG space as

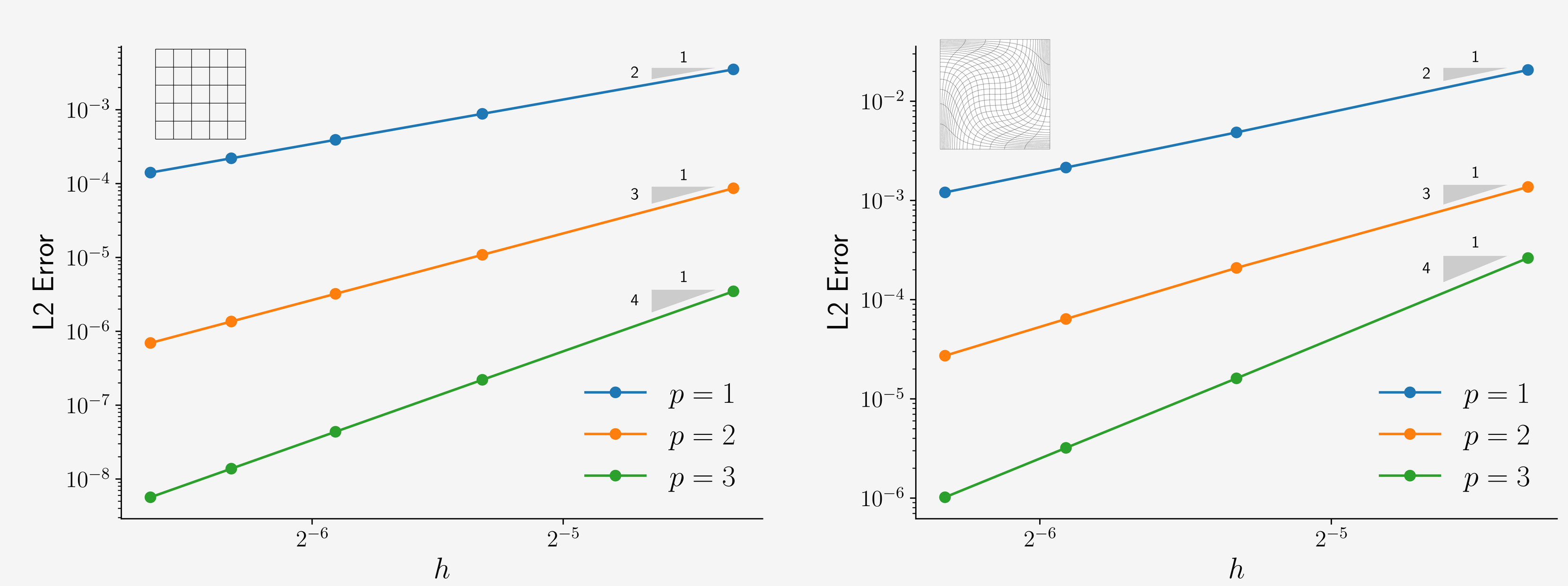
$$\text{DG} = \text{continuous} \bigoplus \text{discontinuous}$$

multigrid — classical smoothers

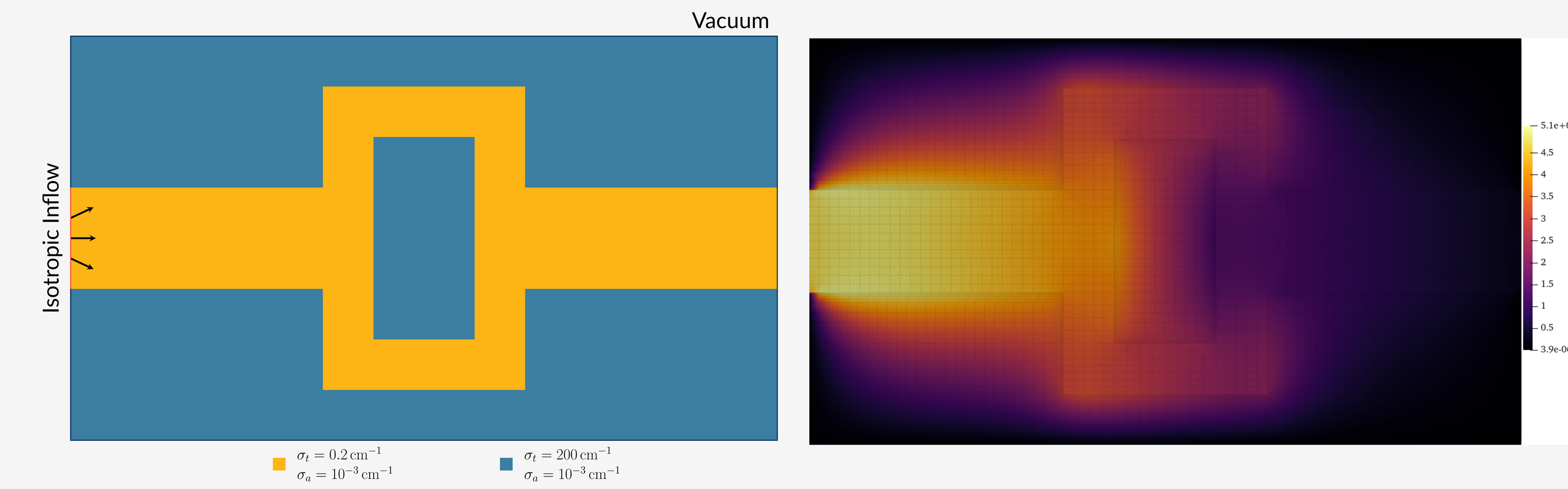
- Combination shown to be effective independent of the mesh size, polynomial order, and penalty parameter for standard diffusion problems
- Continuous operator formed *algebraically* by assembling the original DG system on a continuous finite element space



High-Order Accuracy on Curved Meshes



Iterative Efficiency on Crooked Pipe Problem



- Outer solver: fixed-point with Anderson acceleration, `tol`$=10^{-6}$, 4 Anderson vectors
- Inner solver: BiCGStab, `tol` $= 10^{-7}$, previous outer as initial guess, subspace correction preconditioner (`HypreBoomerAMG`, Jacobi)

	$p=1$		$p=2$		$p=3$	
N_e	Outer	Avg. In	Outer	Avg. In	Outer	Avg. In
256	13	9.38	14	10.71	16	12.88
1024	14	10.64	17	10.18	18	13.5
4096	15	11.0	18	11.17	21	13.38

- Uniform iterations for all orders!

Conclusions

- Designed a DG VEF method that has
  - High-order accuracy
  - Compatibility with curved meshes
  - Efficient preconditioned iterative solvers
- DG VEF shown to be effective on challenging proxy problem from TRT
  - Uniform inner and outer iterations
- Currently working to implement this method in LLNL's next-generation multiphysics code
  - Will be run on El Capitan, one of the world's first exascale computers

References

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