High Order Mixed Finite Element Discretization for the Variable Eddington Factor Equations

Advanced Discretization Techniques for Deterministic Transport I

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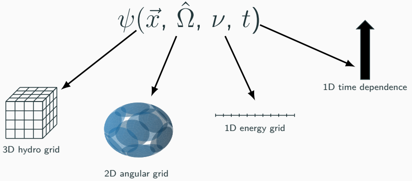
Outline

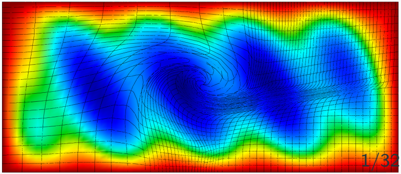
1. Background
2. S_n Discretization
3. VEF Discretization
4. Results

Background

Motivation

- LLNL purusing high-order DG S_n on curved meshes (Haut et al., NSE 2019)
- Raviart-Thomas MFEM successful for radiation diffusion (Maginot and Brunner, JCTT 2018)
- MFEM successful for VEF in 1D (Olivier and Morel, JCTT 2017)
- Anistratov and Warsa showed consistent and independent discretizations perform equally well (NSE 2018, JCTT 2018)
- Can VEF be discretized with methods from radiation diffusion?
    - VEF discretization would match hydro discretization
    - Use established framework for high-order on curved meshes





Variable Eddington Factor Method/Quasidiffusion

- Robust nonlinear transport algorithm
- Two-level in angle
- Nonlinear projective iteration *not* additive correction
- Produces two solutions: one from transport, one from solution of VEF equations
- Consistent Discretization
  - Discretized VEF matches discretized transport exactly
- Inconsistent Discretization
  - Differ by discretization error
  - Acceleration properties the same (if VEF data properly represented)
  - Can match multiphysics discretization
- Flexibility opens door to many new algorithms

Moment Equations

- Steady-state, mono-energetic, isotropic scattering and source:

$$\hat{\Omega} \cdot \nabla \psi + \sigma_t \psi = \frac{\sigma_s}{4\pi} \int \psi \, \mathrm{d}\Omega' + \frac{Q}{4\pi}$$

- Angular moments always have more unknowns than equations

$$\nabla \cdot \vec{J} + \sigma_a \phi = Q$$

$$\nabla \cdot \mathbf{P} + \sigma_t \vec{J} = 0$$

with

$$\phi = \int \psi \, \mathrm{d}\Omega \,, \quad \vec{J} = \int \hat{\Omega} \, \psi \, \mathrm{d}\Omega \,, \quad \mathbf{P} = \int \hat{\Omega} \otimes \hat{\Omega} \, \psi \, \mathrm{d}\Omega$$

- 3D $\Rightarrow 6 + 3 + 1 = 10$ unknowns but only 4 equations

The Philosophy of VEF

- When in doubt, multiply and divide by the scalar flux

$$\mathbf{P} = \int \hat{\Omega} \otimes \hat{\Omega}\, \psi \,\mathrm{d}\Omega \rightarrow \underbrace{\frac{\int \hat{\Omega} \otimes \hat{\Omega}\, \psi \,\mathrm{d}\Omega}{\int \psi \,\mathrm{d}\Omega}}_{\mathbf{E}} \phi = \mathbf{E}\phi$$

- VEF equations:

$$\nabla \cdot \vec{J} + \sigma_a \phi = Q$$
$$\nabla \cdot (\mathbf{E}\phi) + \sigma_t \vec{J} = 0$$

- By contrast, diffusion with tensor coefficient in first order form

$$\nabla \cdot \vec{J} + \sigma_a \phi = Q$$
$$\mathbf{D} \cdot \nabla\phi + \vec{J} = 0$$

- VEF has tensor inside divergence $\Rightarrow$ lots of mixed derivatives, difficult to discretize

Linear Transport VEF Algorithm

- Solve

$$\hat{\Omega} \cdot \nabla \psi^{\ell+1/2} + \sigma_t \psi^{\ell+1/2} = \frac{\sigma_s}{4\pi} \phi^\ell + \frac{Q}{4\pi}$$

  for $\psi^{\ell+1/2}$

- Compute Eddington tensor:

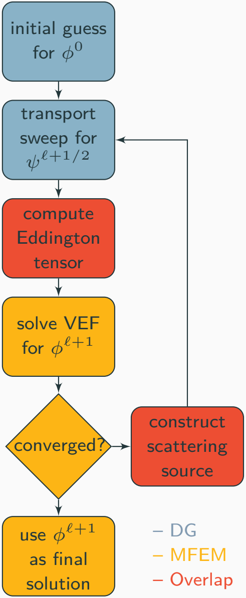
$$\mathbf{E}^{\ell+1/2} = \frac{\int \hat{\Omega} \otimes \hat{\Omega}\, \psi^{\ell+1/2} \, \mathrm{d}\Omega}{\int \psi^{\ell+1/2} \, \mathrm{d}\Omega}$$

- Solve VEF equations for updated scalar flux $\phi^{\ell+1}$

$$\nabla \cdot \vec{J}^{\ell+1} + \sigma_a \phi^{\ell+1} = Q \,,$$

$$\nabla \cdot \left(\mathbf{E}^{\ell+1/2} \phi^{\ell+1} \right) + \sigma_t \vec{J}^{\ell+1} = 0 \,.$$

- Update scattering term directly with VEF solution
- Stop when $\|\phi^{\ell+1} - \phi^\ell\| < tol$



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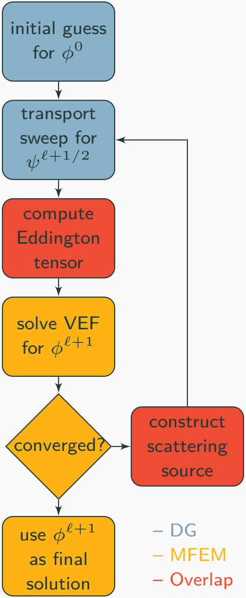


Acceleration Properties

- Acceleration occurs because Eddington tensor converges more rapidly than the scalar flux
  - $\mathbf{E}$ depends on angular shape not magnitude
  - ψ converges quickly in angular shape
  - Compensates lagging scattering term

S_n Discretization

Transport Weak Form

- Multiply by test function u and integrate over a *single* element

$$\int_{\kappa_e} u\,\hat{\Omega}\cdot\nabla\psi\,\mathrm{d}V + \int_{\kappa_e} \sigma_t\,u\psi\,\mathrm{d}V = \frac{1}{4\pi}\int_{\kappa_e} \sigma_s\,u\phi\,\mathrm{d}V + \frac{1}{4\pi}\int_{\kappa_e} u\,Q\,\mathrm{d}V$$

- Integrate by parts since ψ is discontinuously approximated

$$\oint_{\partial\kappa_e} \hat{\Omega}\cdot\hat{n}\,u\widehat{\psi}\,\mathrm{d}A - \int_{\kappa_e} \hat{\Omega}\cdot\nabla u\,\psi\,\mathrm{d}V + \int_{\kappa_e} \sigma_t\,u\psi\,\mathrm{d}V$$
$$= \frac{1}{4\pi}\int_{\kappa_e} \sigma_s\,u\phi\,\mathrm{d}V + \frac{1}{4\pi}\int_{\kappa_e} u\,Q\,\mathrm{d}V$$

where $\widehat{\psi}$ is the *upwind* angular flux

$$\hat{\Omega}\cdot\hat{n}\,\widehat{\psi} = \frac{1}{2}\Big(|\hat{\Omega}\cdot\hat{n}| + \hat{\Omega}\cdot\hat{n}\Big)\,\psi_e + \frac{1}{2}\Big(|\hat{\Omega}\cdot\hat{n}| - \hat{\Omega}\cdot\hat{n}\Big)\,\psi_{e'}$$

where $\hat{n}$ points from element e to element e'

Jumps in Transport

- Shared face between e and e', sum of boundary terms

$$\oint_\Gamma \hat{\Omega} \cdot \hat{n}\, u\psi \,\mathrm{d}A = \underbrace{\oint_\Gamma \hat{\Omega} \cdot \hat{n}\, u_e\psi_e \,\mathrm{d}A}_{\text{Local outflow}} + \underbrace{\oint_\Gamma \hat{\Omega} \cdot \hat{n}'\, u_{e'}\psi_{e'} \,\mathrm{d}A}_{\text{Neighbor's inflow}}$$

- Noting that $\hat{n}' = -\hat{n}$

$$\oint_\Gamma \hat{\Omega} \cdot \hat{n}\, u\psi \,\mathrm{d}A = \oint_\Gamma \hat{\Omega} \cdot \hat{n}[u_e\psi_e - u_{e'}\psi_{e'}] \,\mathrm{d}A$$

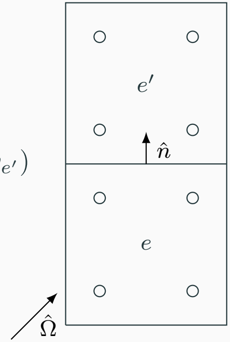
- Jumps and averages identity

$$u_e\psi_e - u_{e'}\psi_{e'} = \frac{1}{2}(u_e - u_{e'})(\psi_e + \psi_{e'}) + \frac{1}{2}(u_e + u_{e'})(\psi_e - \psi_{e'})$$

- Since upwinding is used, $\psi_e = \psi_{e'} = \widehat{\psi}$ for $\vec{x} \in \Gamma$

$$\oint_\Gamma \hat{\Omega} \cdot \hat{n}\, u\psi \,\mathrm{d}A = \oint_\Gamma \hat{\Omega} \cdot \hat{n}[u_e - u_{e'}]\, \widehat{\psi} \,\mathrm{d}A$$

- Jump due to discontinuity of *test function*



FEM Interpolations

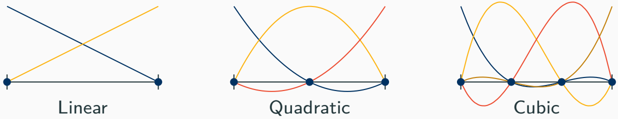
- Let $b_1, \ldots, b_n$ be a set of Lagrange shape functions
- Interpolate ψ and u with linear combination of shape functions

$$\psi(\vec{x}) = \sum_j b_j(\vec{x})\psi_j\,, \quad u(\vec{x}) = \sum_i b_i(\vec{x})u_i$$

- Discrete system for each angle

$$[\mathbf{F} + \mathbf{G} + \mathbf{M}_t]\,\underline{\psi} = \widetilde{\mathbf{M}}_s\underline{\phi} + \mathbf{Q}$$

- Get solution as a *grid function* in each element
- On interior faces, use upwinding $\Rightarrow$ unique definition on entire domain



Computing VEF Tensor

- Use S_n quadrature and FEM interpolation

$$\begin{aligned}\mathbf{E}(\vec{x}) &= \frac{\sum_d \hat{\Omega}_d \otimes \hat{\Omega}_d\, \psi_d(\vec{x}) w_d}{\sum_d \psi_d(\vec{x}) w_d} \\ &= \frac{\sum_d \hat{\Omega}_d \otimes \hat{\Omega}_d\, w_d \sum_j b_j(\vec{x})\psi_{d,j}}{\sum_d w_d \sum_j b_j(\vec{x})\psi_{d,j}} \\ &= \frac{\sum_j b_j(\vec{x}) \sum_d \hat{\Omega}_d \otimes \hat{\Omega}_d\, \psi_{d,j} w_d}{\sum_j b_j(\vec{x}) \sum_d \psi_{d,j} w_d} \\ &= \frac{\sum_j b_j(\vec{x})\mathbf{P}_j}{\sum_j b_j(\vec{x})\phi_j}\end{aligned}$$

- Numerator and denominator are interpolated independently $\Rightarrow$ $\mathbf{E}$ is an *improper rational polynomial* in space
- Store the zeroth and second moments at the nodes and interpolate them independently

VEF Discretization

Weak Form

- VEF Equations

$$\nabla \cdot \vec{J} + \sigma_a \phi = Q$$

$$\nabla \cdot (\mathbf{E}\phi) + \sigma_t \vec{J} = 0$$

- Test zeroth moment with scalar test function u and integrate over *entire domain*

$$\int_\mathcal{D} u \, \nabla \cdot \vec{J} \, \mathrm{d}V + \int_\mathcal{D} \sigma_a \, u\phi \, \mathrm{d}V = \int_\mathcal{D} u \, Q \, \mathrm{d}V$$

- Test first moment with vector-valued test function $\vec{v}$ and integrate over *entire domain*

$$\int_\mathcal{D} \vec{v} \cdot \nabla \cdot (\mathbf{E}\phi) \, \mathrm{d}V + \int_\mathcal{D} \sigma_t \, \vec{v} \cdot \vec{J} \, \mathrm{d}V = 0$$

Required Spaces

- Need all integrals in weak form to be integrable i.e. $\int (\cdot) \, \mathrm{d}V < \infty$
- Terms without derivatives are ok since $\phi \, , \vec{J}$ are physical quantities
- $\mathbf{E}$ is discontinuous, derivatives on both $\vec{J}$ and ϕ is needlessly strong
  $\Rightarrow$ integrate first moment by parts

$$\int_{\mathcal{D}} \vec{v} \cdot \nabla \cdot (\mathbf{E}\phi) \, \mathrm{d}V = \int_{\partial \mathcal{D}} \vec{v} \cdot \mathbf{E} \cdot \hat{n} \, \phi \, \mathrm{d}A - \int_{\mathcal{D}} \nabla \vec{v} : \mathbf{E} \, \phi \, \mathrm{d}V$$

where

$$\nabla \vec{v} = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} \end{bmatrix} \, ,$$

$$\mathbf{A} : \mathbf{B} = \sum_i \sum_j A_{ij} B_{ij} = A_{11} B_{11} + A_{12} B_{12} + A_{21} B_{21} + A_{22} B_{22}$$

- Weakens requirements on u, ϕ, and $\mathbf{E}$

Required Spaces (cont.)

- Now require that

$$\int_{\mathcal{D}} u \, \nabla \cdot \vec{J} \, \mathrm{d}V < \infty$$

$$\int_{\mathcal{D}} \nabla \vec{v} : \mathbf{E} \, \phi \, \mathrm{d}V < \infty$$

- Galerkin FEM $\Rightarrow u\,,\phi$ and $\vec{v}\,,\vec{J}$ lie in the same spaces, respectively
- No derivatives of u or ϕ required $\Rightarrow$ seek $u\,,\phi \in L^2(\mathcal{D})$ (discontinuous FEM)
  - $u \in L^2(\mathcal{D})$ if $\int_{\mathcal{D}} u^2 \, \mathrm{d}V < \infty$
- Gradient of $\vec{v} \Rightarrow$ seek $\vec{v}\,,\vec{J} \in \vec{H}^1(\mathcal{D})$ (continuous FEM)
  - $\vec{v} \in \vec{H}^1(\mathcal{D})$ if $\int_{\mathcal{D}} \vec{v} \cdot \vec{v} \, \mathrm{d}V + \int_{\mathcal{D}} \nabla \vec{v} : \nabla \vec{v} \, \mathrm{d}V < \infty$
  - $H(\mathrm{div})$ not strong enough $\Rightarrow$ Raviart-Thomas FEM won't work

Comparison to Radiation Diffusion

- Weak form of radiation diffusion in first-order form

$$\int_{\mathcal{D}} u \, \nabla \cdot \vec{J} \, \mathrm{d}V + \int_{\mathcal{D}} \sigma_a \, u\phi \, \mathrm{d}V = \int_{\mathcal{D}} u \, Q \, \mathrm{d}V$$

$$\int_{\partial\mathcal{D}} \vec{v} \cdot \hat{n} \, \phi \, \mathrm{d}A - \int_{\mathcal{D}} \nabla \cdot \vec{v} \, \phi \, \mathrm{d}V + 3 \int \sigma_t \, \vec{v} \cdot \vec{J} \, \mathrm{d}V = 0$$

- Boundary term on an interior face + jumps and averages

$$\oint_{\Gamma} \vec{v} \cdot \hat{n} \, \phi \, \mathrm{d}A = \oint_{\Gamma} \frac{1}{2}(\vec{v}_e \cdot \hat{n} - \vec{v}_{e'} \cdot \hat{n})(\phi_e + \phi_{e'}) \, \mathrm{d}A + \oint_{\Gamma} \frac{1}{2}(\vec{v}_e + \vec{v}_{e'}) \cdot \hat{n}(\phi_e - \phi_{e'})$$

- If $\vec{v}$ space chosen such that $\vec{v} \cdot \hat{n}$ is continuous across element faces, first term cancels
- If solution is smooth (as expected from an elliptic PDE), then $\phi_e - \phi_{e'} \approx \mathcal{O}(h^{p+1})$, ignore as consistent error
- Left with no boundary terms on interior faces

Are there jumps in MFEM VEF?

- VEF boundary term

$$\oint_\Gamma \vec{v}\cdot\mathbf{E}\cdot\hat{n}\,\phi\,\mathrm{d}A = \oint_\Gamma \vec{v}_e\cdot\mathbf{E}\cdot\hat{n}\,\phi_e\,\mathrm{d}A - \oint_\Gamma \vec{v}_{e'}\cdot\mathbf{E}\cdot\hat{n}\,\phi_{e'}\,\mathrm{d}A$$

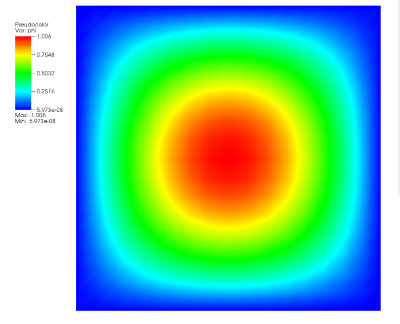
  where $\mathbf{E}$ is computed with the *upwind* angular flux $\Rightarrow$ both elements "agree" on value of $\mathbf{E}$
- Jumps and averages:

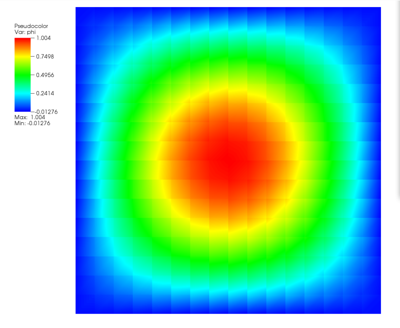
$$\oint_\Gamma \vec{v}\cdot\mathbf{E}\cdot\hat{n}\,\phi\,\mathrm{d}A = \oint_\Gamma \frac{1}{2}(\vec{v}_e\cdot\mathbf{E}\cdot\hat{n} - \vec{v}_{e'}\cdot\mathbf{E}\cdot\hat{n})(\phi_e+\phi_{e'})\,\mathrm{d}A + \oint_\Gamma \frac{1}{2}(\vec{v}_e+\vec{v}_{e'})\cdot\mathbf{E}\cdot\hat{n}(\phi_e-\phi_{e'})\,\mathrm{d}A$$

- Need the $\mathbf{E}\cdot\hat{n}$ component of $\vec{v}$ to be continuous to cancel first term
  - $\mathbf{E}$ rotates and scales the normal
  - First term cancels for $\vec{H}^1(\mathcal{D})$ but not $H(\mathrm{div})$ or $H(\mathrm{curl})$
- Accept jump in ϕ as consistent error $\Rightarrow$ no interior face terms
  - Is this true in transport context?

Initial Try with Raviart-Thomas FEM

Solve VEF with $\mathbf{E} = \frac{1}{3}\mathbf{I}$ and a general $\mathbf{E}$





RT not enough when $\mathbf{E}$ has off-diagonal components

Final Weak Form

- Find $(\phi, \vec{J}) \in L^2(\mathcal{D}) \times \vec{H}^1(\mathcal{D})$ with $\phi|_{\partial\mathcal{D}} = \bar{\phi}$ such that

$$\int_{\mathcal{D}} u \, \nabla \cdot \vec{J} \, \mathrm{d}V + \int_{\mathcal{D}} \sigma_a \, u\phi \, \mathrm{d}V = \int_{\mathcal{D}} u \, Q \, \mathrm{d}V$$

$$-\int_{\mathcal{D}} \nabla \vec{v} : \mathbf{E} \, \phi \, \mathrm{d}V + \int_{\mathcal{D}} \sigma_t \, \vec{v} \cdot \vec{J} \, \mathrm{d}V = -\int_{\partial\mathcal{D}} \vec{v} \cdot \mathbf{E} \cdot \hat{n} \, \bar{\phi} \, \mathrm{d}A$$

  holds for all $(u, \vec{v}) \in L^2(\mathcal{D}) \times \vec{H}^1(\mathcal{D})$
- $\vec{H}^1(\mathcal{D})$ + ignoring jump in $\phi \Rightarrow$ no interior face terms

FEM Interpolation

- Interpolate scalar flux with DG

$$\phi(\vec{x}) = \sum_j B_j(\vec{x})\phi_j\,, \quad B_j \subset L^2(\mathcal{D})$$

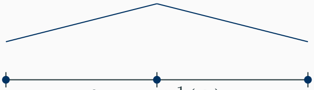
- Interpolate each component of the current with continuous FEM

$$J_d(\vec{x}) = \sum_j N_j(\vec{x})J_{d,j}\,, \quad d = x, y, z\,, \quad N_j \subset H^1(\mathcal{D})$$

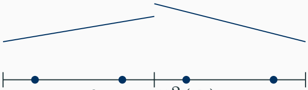
or in vector notation

$$\vec{J}(\vec{x}) = \sum_j \vec{N}_j(\vec{x})J_j\,, \quad \vec{N}_j \subset \vec{H}^1(\mathcal{D})$$

where $\vec{N}_j$ has one component equal to an $H^1(\mathcal{D})$ basis function the rest zero



Linear $H^1(\mathcal{D})$

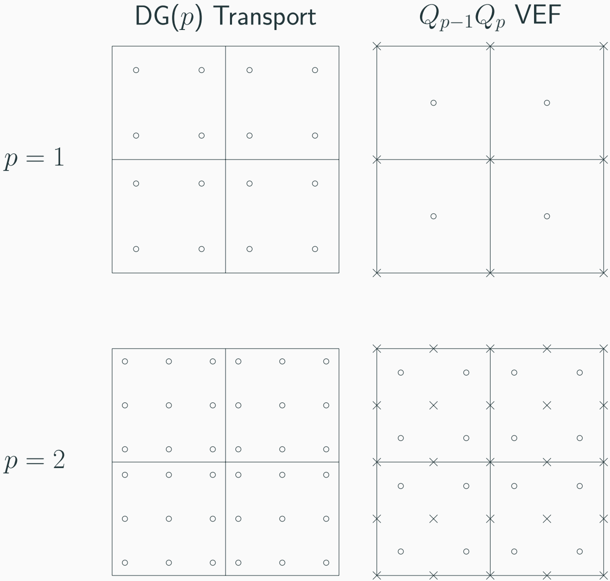


Linear $L^2(\mathcal{D})$

FEM Interpolation (cont.)

- In MFEM literature, it is common to use one order lower polynomial bases for the scalar unknown
  - ϕ constant $+$ $\vec{J}$ linear, ϕ linear $+$ $\vec{J}$ quadratic, etc.
  - $Q_{p-1}Q_p$ matches DG(p)'s $\mathcal{O}(h^{p+1})$
  - Can match transport order of accuracy with VEF ϕ one polynomial order less than ψ
- Mixed form: first moment subject to constraint of zeroth moment
  - Want ratio of discrete equations to match 3:1 ratio of continuous equations
- Found to avoid locking in incompressible solid mechanics
- Not clear if required in this context

$Q_{p-1}Q_p$ Node Placement



Discrete System Leads to Non-Symmetric Saddle Point System

$$\begin{bmatrix} \mathbf{M}_t & -\mathbf{G} \\ \mathbf{D} & \mathbf{M}_a \end{bmatrix} \begin{bmatrix} \underline{J} \\ \underline{\phi} \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \mathbf{f} \end{bmatrix}$$

where

$$[\mathbf{M}_t]_{ij} = \int_{\mathcal{D}} \sigma_t \vec{N}_i \cdot \vec{N}_j \, \mathrm{d}V \,, \quad [\mathbf{G}]_{ij} = \int_{\mathcal{D}} \nabla \vec{N}_i : \mathbf{E} \, B_j \, \mathrm{d}V \,,$$

$$[\mathbf{D}]_{ij} = \int_{\mathcal{D}} B_i \nabla \cdot \vec{N}_j \, \mathrm{d}V \,, \quad [\mathbf{M}_a]_{ij} = \int_{\mathcal{D}} \sigma_a B_i B_j \, \mathrm{d}V \,,$$

$$[\mathbf{g}]_i = -\int_{\partial\mathcal{D}} \vec{N}_i \cdot \mathbf{E} \cdot \hat{n} \, \bar{\phi} \, \mathrm{d}A \,, \quad [\mathbf{f}]_i = \int_{\mathcal{D}} B_i \, Q \, \mathrm{d}V \,.$$

and

$$\mathbf{G} \neq -\mathbf{D}^T$$

due to the presence of the Eddington tensor

Miften-Larsen Transport-Consistent Boundary Conditions

- Manipulate partial currents to get transport consistent inflow current

$$\begin{aligned}
\vec{J} \cdot \hat{n} &= J_n^+ + J_n^- \\
&= (J_n^+ - J_n^-) + 2J_n^- \\
&= \int |\hat{\Omega} \cdot \hat{n}| \, \psi \, \mathrm{d}\Omega + 2J_n^- \\
&= \frac{\int |\hat{\Omega} \cdot \hat{n}| \, \psi \, \mathrm{d}\Omega}{\int \psi \, \mathrm{d}\Omega} \phi + 2J_n^- \\
&= E_b \phi + 2J_n^-
\end{aligned}$$

$$\Rightarrow \bar{\phi} = \frac{1}{E_b} \left[\vec{J} \cdot \hat{n} - 2J_n^- \right]$$

- J_n^- computed from transport boundary conditions
- Adds additional bilinear forms for $\vec{J}$ and ϕ and a linear form to the first moment

Solution Process: Schur Complement

$$\begin{bmatrix} \mathbf{M}_t & -\mathbf{G} \\ \mathbf{D} & \mathbf{M}_a \end{bmatrix} \begin{bmatrix} \underline{J} \\ \underline{\phi} \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \mathbf{f} \end{bmatrix}$$

- Since ϕ is discontinuously approximated, $\mathbf{M}_a$ is block-diagonal
  - Invert blocks independently $\Rightarrow$ can directly invert and store efficiently without fill-in
- Solve zeroth moment for $\underline{\phi}$

$$\mathbf{D}\underline{J} + \mathbf{M}_a\underline{\phi} = \mathbf{f} \Rightarrow \underline{\phi} = \mathbf{M}_a^{-1}[\mathbf{f} - \mathbf{D}\underline{J}]$$

- Solve first moment using $\underline{\phi}$

$$\mathbf{M}_t\underline{J} - \mathbf{G}\underline{\phi} = \mathbf{g} \Rightarrow \underbrace{\left[\mathbf{M}_t + \mathbf{G}\mathbf{M}_a^{-1}\mathbf{D}\right]}_{\mathbf{S}}\underline{J} = \mathbf{g} + \mathbf{G}\mathbf{M}_a^{-1}\mathbf{D}\mathbf{f}$$

- Just need to solve a system of the current unknowns, $\underline{\phi}$ found through matrix multiplications
- $\mathbf{S}$ still non-symmetric, difficult to solve but smaller than original system

Computing the Scattering Term

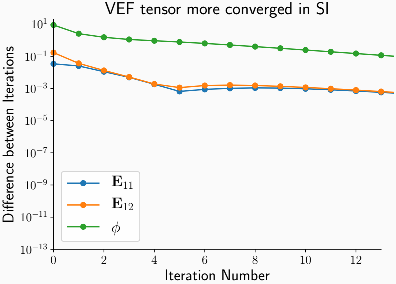
- Once $\underline{\phi}$ is known, the transport scattering mass matrix $\widetilde{\mathbf{M}}_s$ is computed as
$$\left[\widetilde{\mathbf{M}}_s\right]_{ij} = \frac{1}{4\pi}\int \sigma_s b_i B_j \,\mathrm{d}V$$
- Uses interpolations provided by FEM
- $\widetilde{\mathbf{M}}_s\underline{\phi}$ forms the scattering source completing a VEF iteration
- This handles using different polynomial orders for ψ and VEF ϕ
- Equivalent to “direct” mapping in Warsa and Anistratov (JCTT, 2018) when polynomial orders are the same
  - Shown to preserve transport’s order of accuracy

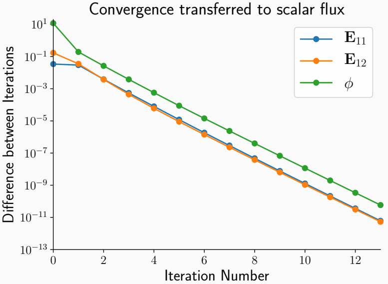
Results

Problem Setup

- Methods implemented in MFEM finite element library
  - All integrals performed numerically with appropriate order Gauss quadrature
- 1 cm $\times$ 1 cm box discretized with uniform quadrilaterals
- S_8 level symmetric angular quadrature
- Iterative tolerance of 10^{-10}
- Unless otherwise noted:
  - Domain is 10 mfp thick with $c = 0.99$
  - Transport solved with $p = 2$ and $Q_1 Q_2$ VEF

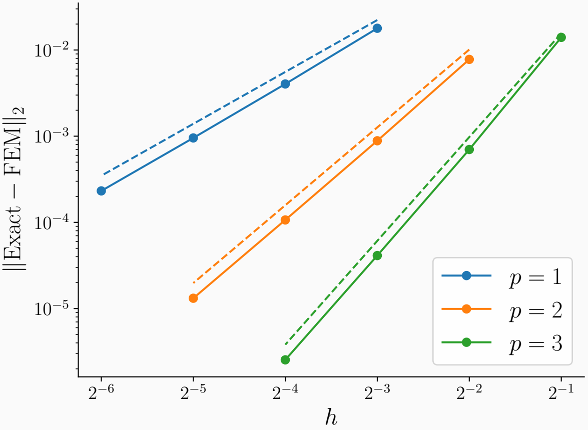
VEF Transfer Fast Rate of Converge to Scalar Flux





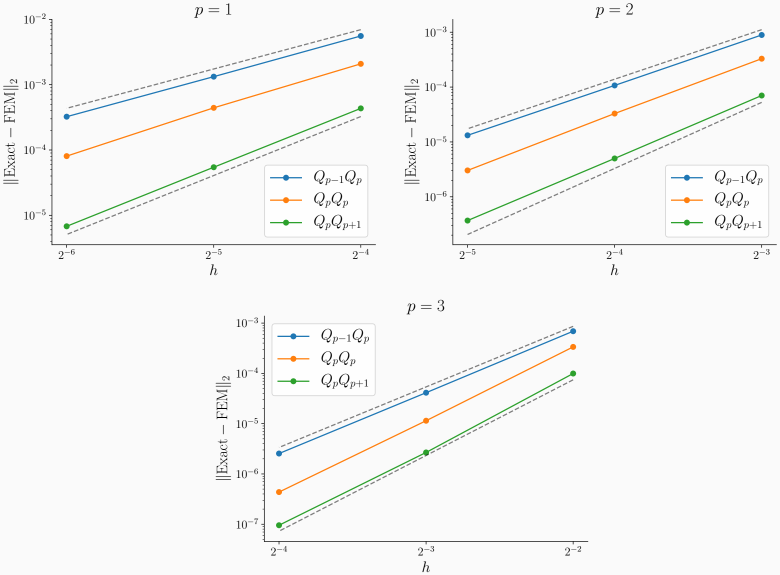
Method of Manufactured Solutions

- MMS solution: $\psi_d = \left(1+\mu_d^2\right)\sin\left(\pi\frac{x+\alpha}{L+2\alpha}\right)\sin\left(\pi\frac{y+\alpha}{L+2\alpha}\right)$
- $\alpha = 0.1$ tests inflow boundary conditions
- Transport: DG(p), VEF: $Q_{p-1}Q_p$, expect $\mathcal{O}(h^{p+1})$



MFEM VEF maintains transport order of accuracy for smooth problems

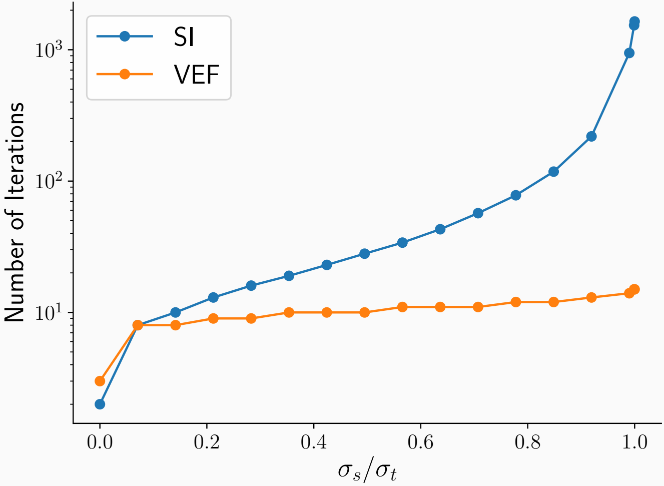
Method of Manufactured Solutions (cont.)



All variations stable, VEF can elevate transport order of accuracy

Scattering Ratio Test

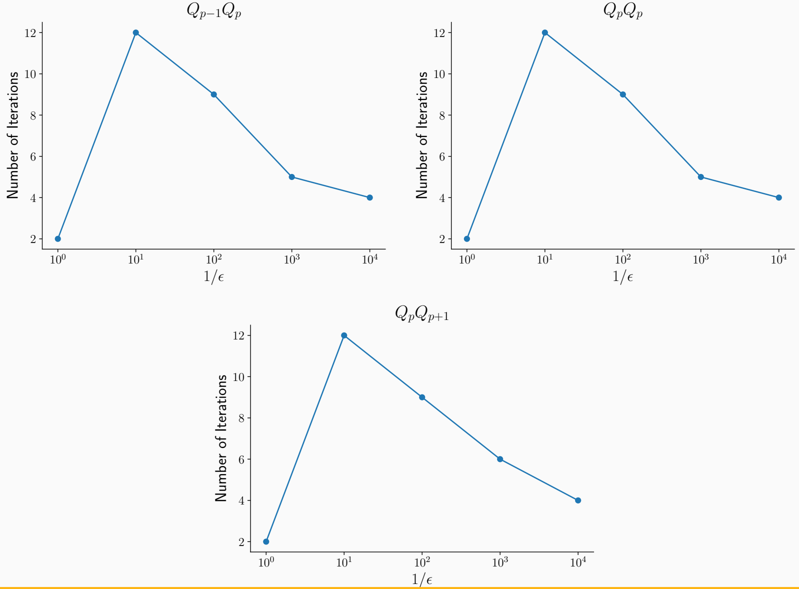
$$\sigma_t = 10\,\text{cm}^{-1}\,, \quad \sigma_s = c\sigma_t\,, \quad Q = 1\,\text{cm}^{-2}\,\text{s}^{-1}\,, \quad p = 2$$



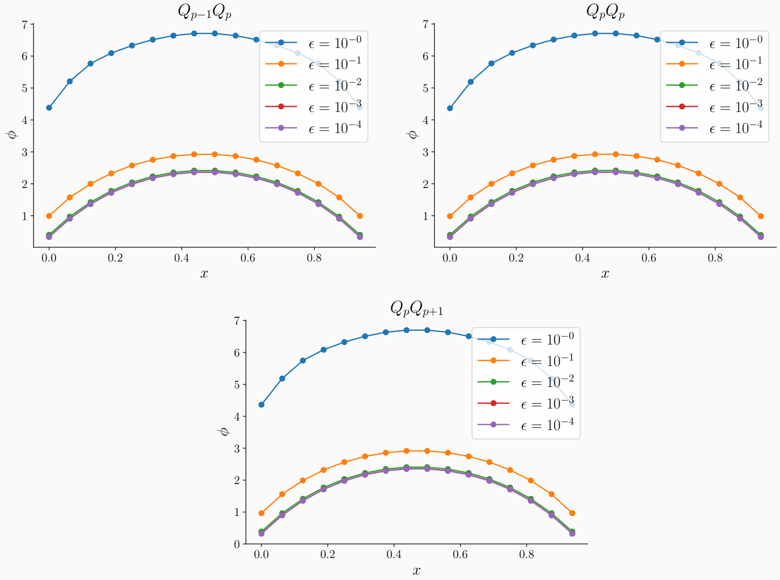
Independent discretization converges rapidly

Thick Diffusion Limit

$$\sigma_t = \frac{1}{\epsilon}\,, \quad \sigma_s = \frac{1}{\epsilon} - \epsilon\,, \quad Q = \epsilon\,, \quad p = 2$$



Thick Diffusion Limit (cont.)



All get diffusion solution

Conclusions

- All combinations of VEF polynomial orders:
  - Maintained or elevated transport order of accuracy
  - Equally accelerated source iteration
  - Had the thick diffusion limit
- Future Work
  - Efficient iterative solver for VEF system
    - MFEM hybdrization is promising
    - Leads to non-symmetric, positive-definite system
  - Run more realistic problems
    - Investigate required quadrature rules for rational polynomial terms
    - Discontinuous solutions to test if jump term needed
  - TRT

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Questions?

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